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# ULTIMATE ADMISSIBLE DYNAMIC STRAINS IN CLOSED CYLINDRICAL VESSELS

**Yu. V. Nemirovskii**

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*A problem of determining the ultimate dynamic state of multilayer closed cylindrical vessels in emergency situations, such as explosive loading by high-intensity internal pressure, is considered. Elastic strains are assumed to be negligibly small as compared to plastic strains; therefore, the problem solution is constructed on the basis of the model of a rigid-plastic material with linear hardening. It is demonstrated that the solution of the dynamic deformation problem considered reduces to integration of a system of two ordinary equations for the functions of displacements of the inner surface of the vessel and of the massive non-deformable cover of the vessel.*

**Key words:** *ultimate admissible dynamic state, plasticity, linear hardening, incompressibility, differential equations, Cauchy problem, model of a rigid-plastic material.*

High-pressure closed vessels are widely used in chemical industry, oil refining, microbiology, rocket engineering, aircraft and ship building, and nuclear propulsion. These vessels usually have a cylindrical or spherical shape and operate in a quasistatic or dynamic mode with high-amplitude pressure oscillations (several thousands atmospheres) at temperatures of the order of 600–800°C. As the high-pressure vessel size is increased, the energy of compressed gases accumulated there drastically increases, which may lead to a catastrophe in emergency situations such as "explosive loading." It is next to impossible and cost-inefficient to create protective shelters for significant reduction of the catastrophe consequences. For this reason, it is important to study the ultimate strain of such vessels and to increase the safety of using such vessels with allowance for possible occurrence of emergency situations. The possibility of replacing single-block vessels by multilayer vessels has been studied for a long time. The use of multilayer structures is promising from the viewpoint of both cost efficiency and operation safety, because it allows using different materials and localization of defects in the layers, resulting in viscous fracture of structures without spalling. Quasistatic deformation of multilayer vessels made of elastic and viscoelastic materials and problems of their rational design in these modes were studied in detail in [1]. A review of investigations of quasistatic deformation of vessels made of perfectly plastic inhomogeneous materials can be found in [2]. A dynamic problem for multilayer vessels made of perfectly plastic materials was considered in [3]. Solving this problem, however, does not allow determining the ultimate admissible strain of a multilayer vessel and the maximum admissible amplitude of dynamic loading corresponding to the pre-fracture state.

An approach that makes it possible to take better account of the real properties of plastic resistance of each material and to study the ultimate dynamic state of pre-fracture of multilayer vessels is considered in the present paper.

Let us consider a multilayer cylindrical vessel with rigidly connected layers and massive non-deformable end covers under conditions of axisymmetric dynamic loading: $p(t) = p_0\varphi(t)$. For certainty, we restrict ourselves to "explosive-type" loads. For instance, we can assume that

$$\varphi(t) = \mathrm{e}^{-\alpha t}$$

Khristianovich Institute of Theoretical and Applied Mechanics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090; nemirov@itam.nsc.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 51, No. 4, pp. 183–187, July–August, 2010. Original article submitted March 12, 2010.

 

($t$ is the time). We also assume that the problem of heat conduction for the multilayer cylinder is solved independently, under the assumption of a constant temperature along the vessel centerline [4]. As plastic strains in the pre-fracture state are several orders higher than elastic strains, the latter are neglected, and the model of a rigid-plastic materials with linear hardening in the form of Hencky–Ilyushin relations [5] is used. Taking into account that the material in the plastic state is incompressible, we obtain the following equality for the $i$th layer of the cylinder:

$$\varepsilon_{1i} + \varepsilon_{2i} + \varepsilon_3 = 3\varepsilon_{iT}. \tag{1}$$

In this equation, we have

$$\varepsilon_{1i} = \frac{\partial \bar{u}_i}{\partial \bar{r}}, \qquad \varepsilon_{2i} = \frac{\bar{u}_i}{\bar{r}}, \qquad \varepsilon_3 = \frac{\partial \bar{v}}{\partial \bar{z}}, \qquad \varepsilon_{iT} = \int\limits_{\bar{T}_0}^{\bar{T}_i} \bar{\alpha}_i(\bar{T})\bar{T}\, d\bar{T},$$

$\bar{u}_i$ and $\bar{v}$ are the radial and axial displacements, $\bar{r}$ and $\bar{z}$ are the radial and axial coordinates, and $\bar{T}_i$ and $\bar{\alpha}_i$ are the temperature and the temperature coefficient of linear expansion in the $i$th layer, respectively. Further, we assume that the temperature does not change much in the course of loading; hence, $\bar{T}_i = \bar{T}_i(\bar{r})$ and $\varepsilon_{iT} = \varepsilon_{iT}(\bar{r})$.

We introduce the dimensionless quantities

$$x = \frac{\bar{r}}{\bar{r}_0}, \qquad u_i = \frac{\bar{u}_i}{\bar{r}_0}, \qquad v = \frac{\bar{v}}{L}, \qquad z = \frac{\bar{z}}{L}, \qquad a_i = \frac{\bar{r}_i}{\bar{r}_0}.$$

Then, Eq. (1) takes the form

$$\frac{\partial}{\partial x}(xu_i) = 3\varepsilon_{iT}(x) - \varepsilon_3(t)x, \tag{2}$$

where

$$\varepsilon_{1i} = \frac{\partial u_i}{\partial x}, \qquad \varepsilon_{2i} = \frac{u_i}{x}, \qquad \varepsilon_{3i} = \varepsilon_3(t) = \frac{\partial v}{\partial z}.$$

Integrating Eq. (2), we obtain

$$u_i(x,t) = \frac{a_{i-1}}{x}\, w_i(t) + \varphi_i(x) - \frac{x^2 - a_{i-1}^2}{2x}\, \varepsilon_3(t), \tag{3}$$

where

$$\varphi_i(x) = \frac{1}{x} \int\limits_{a_{i-1}}^{x} \varepsilon_{iT} x\, dx \qquad (i = 1, 2, \ldots, n),$$

and $w_i(t)$ is the displacement of the inner boundary of the $i$th layer.

Using the continuity conditions $u_i(a_i, t) = u_{i+1}(a_i, t)$, we can write equality (3) in the form

$$a_{i-1} w_i(t) = a_0 w_1(t) + b_i \varepsilon_3(t) + c_i,$$

where

$$b_i = \frac{a_{i-1}^2 - a_0^2}{2}, \qquad c_i = \sum_{j=0}^{i-1} a_j \varphi_j(a_j), \qquad \varphi_0(a_0) = 0.$$

Then, the relations for the strains in the $i$th layer are

$$\varepsilon_{2i} = \frac{1}{x^2}\Big(a_0 w_1(t) - \frac{x^2 - a_0^2}{2}\, \varepsilon_3(t) + c_i + x\varphi_i(x)\Big), \qquad \varepsilon_{1i} = 3\varepsilon_{iT} - \varepsilon_{2i} - \varepsilon_3(t),$$

while the intensity of the strains in the $i$th layer is

$$e_i = \frac{\sqrt{2}}{3}\left[(\varepsilon_{1i} - \varepsilon_{2i})^2 + (\varepsilon_{2i} - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_{1i})^2\right]^{1/2} = \frac{\sqrt{2}}{3}\left[3\varepsilon_{iT}(\varepsilon_{iT} - \varepsilon_{2i} - \varepsilon_3) + (\varepsilon_{2i} + \varepsilon_3)^2 - \varepsilon_{2i}\varepsilon_3\right]^{1/2}.$$

Using the law of deformation of the material in the $i$th layer, we find ($a_{i-1} \leqslant x \leqslant a_i$)

$$\sigma_{2i} - \sigma_{1i} = (2/3)(\sigma_{0i} e_i^{-1} + \gamma_i)(\varepsilon_{2i} - \varepsilon_{1i}); \qquad (4)$$

$$\sigma_{3i} = \sigma_{1i} + (2/3)(\sigma_{0i} e_i^{-1} + \gamma_i)(\varepsilon_3(t) - \varepsilon_{1i}), \qquad (5)$$

where $\sigma_{ki}$ ($k = 1, 2, 3$), $\sigma_{0i}$, and $\gamma_i$ are the dimensionless stresses, yield stress, and hardening modulus, respectively.

The equation of motion of the material in the $i$th layer has the form

$$\frac{\partial \sigma_{1i}}{\partial x} + \frac{\sigma_{1i} - \sigma_{2i}}{x} = \rho_i \ddot{u}_i, \qquad (6)$$

where the dot indicates the partial derivative with respect to the dimensionless time ($\tau = t/t_0$). Substituting Eqs. (3) and (4) into Eq. (6) and integrating the resultant equation with respect to $x$ from $a_{i-1}$ to $x$, we find

$$\sigma_{1i}(x, t) = -q_i(t) + \Phi_{1i}(x, w_1, \varepsilon_3) + \Phi_{2i}(x)\ddot{w}_1 + \Phi_{3i}(x)\ddot{\varepsilon}_3 \qquad (i = 1, 2, \ldots, n). \qquad (7)$$

Here, we have

$$\Phi_{1i}(x, w_1, \varepsilon_3) = \frac{2}{3} \int\limits_{a_{i-1}}^{x} \frac{\sigma_{0i} e_i^{-1} + \gamma_i}{x} (\varepsilon_{2i} - \varepsilon_{1i})\, dx,$$

$$\Phi_{2i}(x) = a_0 \rho_i \ln \frac{x}{a_{i-1}}, \qquad \Phi_{3i}(x) = \frac{\rho_i}{2} \int\limits_{a_{i-1}}^{x} \frac{x^2 + a_0^2}{x}\, dx.$$

Similarly, for the $(i+1)$th layer, we obtain

$$\sigma_{1(i+1)}(x, t) = -q_{i+1}(t) + \Phi_{1(i+1)}(x, w_1, \varepsilon_3) + \Phi_{2(i+1)}(x) + \Phi_{3(i+1)}(x). \qquad (8)$$

Using the conditions of continuity of the radial stresses $\sigma_{1i+1}(a_i, t) = \sigma_{1i}(a_i, t)$, we find from Eqs. (7) and (8) that

$$q_{i+1}(t) = -q_i(t) + \Phi_{1i}(x, w_1, \varepsilon_3) + \Phi_{2i}(a_i)\ddot{w}_1 + \Phi_{3i}(a_i)\ddot{\varepsilon}_3 \qquad (i = 1, 2, \ldots, n), \qquad (9)$$

where

$$q_i(t) = q_1(t) - \sum_{k=1}^{i-1} \Phi_{1k}(a_k, w_1, \varepsilon_3) - \Big(\sum_{k=1}^{i-1} \Phi_{2k}(a_k)\Big)\ddot{w}_1 - \Big(\sum_{k=1}^{i-1} \Phi_{3k}(a_k)\Big)\ddot{\varepsilon}_3.$$

Taking into account that $q_1(t) = -p_0\varphi(t)$ and $q_{n+1}(t) = 0$, we obtain an equation that relates the functions $\varepsilon_3(t)$ and $w_1(t)$:

$$p_0\varphi(t) = \Big(\sum_{k=1}^{n} \Phi_{2k}(a_k)\Big)\ddot{w}_1 + \Big(\sum_{k=1}^{n} \Phi_{3k}(a_k)\Big)\ddot{\varepsilon}_3 + \sum_{k=1}^{n} \Phi_{1k}(a_k). \qquad (10)$$

We write the equation of motion of the vessel cover in the dimensionless form as

$$M\ddot{\varepsilon}_3 = p_0\varphi(t) - \sum_{i=1}^{n} \int\limits_{a_{i-1}}^{a_i} \sigma_{3i} x\, dx, \qquad M = \frac{\bar{M}\bar{l}t_0^2}{\pi\sigma_0\bar{r}_0^2}, \qquad (11)$$

where $\bar{M}$ is the cover mass, $\bar{r}_0$ and $\bar{l}$ are the radius of the inner cavity and the half-length of the cylindrical vessel, and $t_0$ and $\sigma_0$ are the scaling parameters of time and stresses. Taking into account Eqs. (5) and (9), we can bring Eq. (11) to the form

$$A_1\ddot{w}_1 + A_2\ddot{\varepsilon}_3 - \Psi_1(\varepsilon_3, w_1) = p_0\Big(1 - \frac{a_n^2 - a_0^2}{2}\Big)\varphi(t), \qquad (12)$$

where

$$A_1 = -\sum_{i=1}^{n}\Big[\frac{a_i^2 - a_{i-1}^2}{2}\Big(\sum_{k=1}^{i-1}\Phi_{2k}(a_k)\Big) + \int_{a_{i-1}}^{a_i}\Phi_{2i}(x)x\,dx\Big],$$

$$A_2 = M - \sum_{i=1}^{n}\Big[\frac{a_i^2 - a_{i-1}^2}{2}\Big(\sum_{k=1}^{i-1}\Phi_{3k}(a_k)\Big) + \int_{a_{i-1}}^{a_i}\Phi_{3i}(x)x\,dx\Big],$$

$$\Psi_1(\varepsilon_3, w_1) = \sum_{i=1}^{n}\Big[\Big(\frac{a_i^2 - a_{i-1}^2}{2}\sum_{k=1}^{i-1}\Phi_{1k}(a_k, w_1, \varepsilon_3)\Big) - \frac{2}{3}\int_{a_{i-1}}^{a_i}[(\sigma_{0i}e_i^{-1} + \gamma_i)(\varepsilon_3 - \varepsilon_{1i})]x\,dx\Big].$$

Thus, solving the problem considered reduces to integrating a system of two ordinary differential equations (10) and (12) with respect to two functions $w_1(t)$ and $\varepsilon_3(t)$. The initial conditions for this system have the form

$$\dot{w}_1(0) = \dot{\varepsilon}_3(0) = w_1(0) = \varepsilon_3(0) = 0.$$

The solution of the problem considered should be sought at amplitudes higher than the ultimate plastic pressure $p_0^0$ for perfectly plastic materials, which is determined by the method described in [3]. At pressure amplitudes $p_0 < p_0^0$, the vessel has an increased safety margin. Therefore, we consider the cases with $p_0^0 < p_0 \leqslant p^*$.

To find the ultimate admissible upper limit of pressure, we use the area $D_*$ under the strain diagram up to the point corresponding to the ultimate stress limit $\sigma_p^*$ as a criterion of fracture for the material considered. In this case, using the linear hardening law, we obtain

$$D_* = (\sigma_0 + \sigma_p)e^*/2.$$

The equality $e_i(a_{i-1}, t_i^*) = e_i^*$ allows us to determine the time $t_i^*$ of fracture occurrence in the $i$th layer. By requiring the condition $\dot{w}_1(t_i^*) = 0$ to be satisfied, we can determine the ultimate admissible amplitude $p_{0i}^*$. Then, $p^*$ is found from the condition

$$p^* = \min(p_{0i}^*) \qquad (i = 1, 2, \ldots, n).$$

Exceeding this value of the pressure amplitude leads to fracture in one of the layers. The structure as a whole is not destroyed, but reliability of the vessel is drastically deteriorated. The present paper is not aimed at analyzing the fracture zone evolution; therefore, such an analysis is not conducted here.

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